MA’RUZA

FUNKSIYA HOSILASI

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Tayanch so’z va iboralar: hosila; urinma; normal; differensial; differensiallash; murakkab funksiya; oshkormas funksiya; teskari funksiya; parametrik funksiya.

1. Hosila tushunchasi

$y=f(x)$ funksiya $x\in D$ nuqtaning biror atrofida aniqlangan va uzluksiz bo’lsin. x argumentga $(x+\Delta x)\in D$ shartni qanoatlantiradigan Δx orttirma beramiz, bu holda funksiyaning tegishli ortirmasi

$$\Delta y=f(x+\Delta x)-f(x)$$

bo’ladi.

Funksiya hosilasining ta’rifi

***Ta’rif*:** *$y=f(x)$ funksiyaning x bo’yicha hosilasi deb, funksiyaning x nuqtadagi orttirmasi Δy ni argument orttirmasi Δx ga nisbatining Δx nolga intilgandagi limitiga aytiladi:*

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Funksiyaning har xil x nuqtalardagi hosilasini topish mumkin, ya’ni funksiya hosilasi ham x ning funksiyasi buladi. Bunda hosilaning aniq-lanish D' sohasi funksiyaning uzluksizlik sohasi D ga tegishli bo’ladi:

$D'\subset D$. Ixtiyoriy x nuqtadagi hosila y'_x, $\frac{dy}{dx}$, $f'(x)$, $\frac{df(x)}{dx}$ belgilarning biri bilan belgilanadi. Hosilani topish amali differensi-allash deyiladi.

Agar $f'(x)$ hosila mavjud bo’lsa, $f(x)$ funksiya x nuqtada differen-siallanuvchi deyiladi. Biror oraliqning har bir nuqtasida differen-siallanuvchi funksiya, shu oraliqda differensiallanuvchi deyiladi. Agar $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\infty$ bo’lsa, $y=f(x)$ funksiya x nuqtada cheksiz hosilaga ega deyiladi.

Hosila ta’rifini misollar yechishga tadbiq qilish algoritmi

1. $y=f(x)$ funksiyaning $[x,x+\Delta x]$ oraliqdagi orttirmasi hisoblanadi :

   $\Delta y=f(x+\Delta x)-f(x)$

2. Funksiyaning orttirmasi argument orttirmasiga bo’linadi:

   $$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

3. Keyingi tenglikda Δx ni nolga intiltirib limitga o’tiladi:

   $$y'=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

1-misol

$y = 2x^2 - 3x$ funksiyaning hosilasi topilsin.

Yechilishi:

1) $\Delta y = 2(x+\Delta x)^2 - 3(x+\Delta x) - (2x^2 - 3x);\quad \Delta y = (4x - 3 + 2\Delta x)\cdot \Delta x;$

2) $\dfrac{\Delta y}{\Delta x} = (4x-3) + 2\cdot\Delta x;$

3) $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}(4x - 3 + 2\cdot\Delta x) = 4x - 3;\qquad y' = 4x - 3.$

2-misol

$y = \sin ax, (a$ - son) funksiyaning hosilasi topilsin.

Yechilishi:

1) $\Delta y = \sin a(x+\Delta x) - \sin ax = 2\cos\dfrac{2ax + a\Delta x}{2}\cdot\sin\dfrac{a\cdot\Delta x}{2};$

2) $\dfrac{\Delta y}{\Delta x} = \dfrac{2\cos\dfrac{2ax+a\Delta x}{2}\cdot\sin\dfrac{a\cdot\Delta x}{2}}{\Delta x};$

3) $\lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}\cos\dfrac{2ax+a\Delta x}{2}\cdot\dfrac{a\cdot\sin\dfrac{a\cdot\Delta x}{2}}{\dfrac{a\cdot\Delta x}{2}} = a\lim\limits_{\Delta x\to 0}\cos(ax + \dfrac{a\Delta x}{2})\cdot\lim\limits_{\Delta x\to 0}\dfrac{\sin\dfrac{a\Delta x}{2}}{\dfrac{a\Delta x}{2}} =$

$= a\cos ax;$

bu yerda,

$\lim\limits_{\Delta x\to 0}\cos(ax + \dfrac{a\Delta x}{2}) = \cos ax;\quad \lim\limits_{\Delta x\to 0}\dfrac{\sin\dfrac{a\Delta x}{2}}{\dfrac{a\cdot\Delta x}{2}} = 1.$ Shunday kilib, $(\sin ax)' = a\cos ax.$

2. Hosilaning geometrik ma’nosi. Urinma va normalning tenglamasi

L egri chiziq *y*= *f (x)* tenglama bilan berilgan, bunda *f* (x) biror intervalda aniqlangan va uzluksiz funksiya bo’lsin.

Bu egri chiziqda $M_0(x_0;y_0)$ nuqtani belgilab olamiz. *M(x;y)* egri chiziqning ixtiyoriy nuqtasi bo’lsin. M_0 va *M* nuqtalardan $\overline{M_0M}$ kesuvchi o’tkazamiz.
(*1-shakl*)

<u>Ta’rif:</u> *M nuqta egri chiziq buylab M_0 nuqtaga intilganda $\overline{M_0M}$ kesuvchining limitik holati L egri chiziqqa M_0 nuqtada o’tkazilgan urinma deyiladi*.

Agar $x - x_0 = \Delta x$, $(x = x_0 + \Delta x)$, $\Delta y = f(x_0 + \Delta x - f(x_0)$ deb olinsa, $tg\beta = \dfrac{\Delta y}{\Delta x}$ bo’ladi. $M \to M_0$ da, ya’ni $\Delta x \to 0$ da, funksiyaning uzluksizligiga asosan, $\Delta y \to 0$ ga va kesuvchi urinmaga chegaralanmagan holda intiladi, ya’ni $\lim\limits_{\Delta x\to 0}\beta = \alpha$ bo’ladi.

Demak, $\lim\limits_{\Delta x\to 0} tg\beta = tg\alpha$.

Shu sababli, urinma burchak koeffisenti $k_y = tg\alpha = \lim\limits_{\Delta x\to 0} tg\beta = \lim\limits_{\Delta x\to 0}\dfrac{\Delta y}{\Delta x} = f'(x_0)$ bo’ladi.

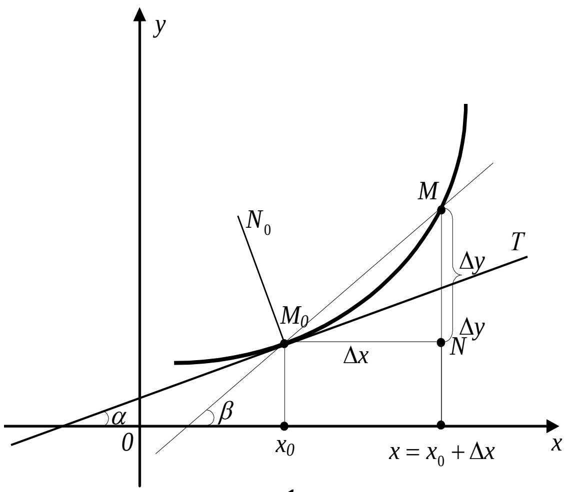
<u>Xulosa:</u> *Funksiya grafigiga x_0 absissali nuqtadan o’tkazilgan urinmaning burchak koeffisenti bu funksiya hosilasining x_0 nuqtadagi qiymatiga teng:*

$$k_y = f'(x_0).$$

(bu holat hosilaning geometrik ma'nosini ifodalaydi).

Agar nuqtadagi hosila chekli bo'lsa, u holda urinma *Ox* o'qining musbat yo'nalishi bilan $\alpha \neq \frac{\pi}{2}$ burchak hosil qiladi. $f'(x_0) = \infty$ bo'lgan holda esa urinma *Ox* o'qi bilan $\alpha = \frac{\pi}{2}$ (to'g'ri) burchak hosil qiladi.

Endi *L* egri chiziqka $M_0(x_0;y_0)$ nuqtada o'tkazilgan urinma tenglamasini tuzamiz. Urinma to'g'ri chiziq $M_0(x_0;y_0)$ nuqtadan o'tgani va $k_y = f'(x_0)$ uning burchak koeffisenti bo'lgani uchun uning tenglamasi $y-y_0 = f'(x_0)(x-x_0)$ ko'rinishga ega bo'ladi (be*rilgan nuqtadan berilgan yunalishda o'tuvchi to'g'ri chiziq tenglamasiga asosan*).



1-шакл

<u>Ta'rif:</u> *Egri chiziq M_0 nuqtaga o'tkazilgan normal deb, M_0 nuqtada o'tgazilgan va (M_0 T) urinmaga tik bo'lgan (M_0 N_0) to'g'ri chiziqka aytiladi.*
(1-shaklga qaralsin).

Normalning burchak koeffisiyenti:

$$k_n = -\frac{1}{k_y} = -\frac{1}{f'(x_0)}$$

Normalning tenglamasi:

$$y - y_0 = k_n(x - x_0); \text{ yoki } y - y_0 = -\frac{1}{f'(x_0)}(x - x_0);$$

Misol

$y = 2x^2$ parabolaga uning $M_0(1;1)$ nuqtasidan o'tkazilgan urinma va nor-malning tenglamasini tuzing.

Yechilishi: $y = 2x^2$ funksiyaning $x_0 = 1$ nuqtadagi hosilasini topamiz:

$y' = 4x$, bu yerdan $y'(1) = 4x\,/_{x=1} = 4$,

ya'ni $f'(x_0) = f'(1) = 4$.

Urinmaning izlanayotgan tenglamasi

$y - 1 = 4(x - 1)$ yoki $4x - y - 3 = 0$.

Normalning tenglamasi:

$y - 1 = -\frac{1}{4}(x - 1)$ yoki $x + 4y - 5 = 0$.

3. Hosilaning fizikaviy ma'nosi

Moddiy M nuqta $s = f(t)$ qonun bilan to'g'ri chiziqli harakat qilsin va Δt vaqtda yo'lning Δs qismini o'tsin. Bu holda $\frac{\Delta s}{\Delta t}$ nisbat harakatdagi nuqtaning Δt vaqtdagi o'rtacha tezligini beradi.

Funksiya orttirmasining argument orttirmasiga nisbati - urtacha tezlikdir

O'rtacha tezlikning $\Delta t \to 0$ dagi limiti t momentdagi oniy tezlik deyiladi, ya'ni

$$v = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta t}$$

Hosila – bu tezlik

<u>Xulosa:</u> *Moddiy nuqtaning t momentdagi to'g'ri chiziqli harakati tezligi deb, s yo'ldan t vaqt bo'yicha olingan hosilaga aytiladi.*

Shunga o'xshash ravishda, tezlanish tezlikdan vaqt bo'yicha olingan hosilaga tengligini ko'rsatish mumkin: $a = v'(t)$.

To'g'ri chiziqli sterjenning x nuqtasidagi chiziqli zichligi massasining x uzunlik bo'yicha hosilasiga teng:

Jismning issiqlik sig'imi issiqlik miqdori Q dan T temperatura bo'yicha olingan hosiladir, ya'ni

$$S = Q'(T)$$

Keltirilgan misollarning barchasi, funksiya o'zgarish tezligining xususiy hollaridir. Shu sababli, o'zgaruvchilarning asl ma'nosi inobatga olinmasa, quyidagini tasdiqlash mumkin:

$y = f(x)$ funksiyaning o'zgarish tezligi y dan x bo'yicha olingan hosiladir.

Hosila – funksiyaning uzgarish tezligidir

4. Hosilaning iqtisodiy ma'nosi

Korxona bir jinsli maxsulot ishlab chiqarsin. Bu holda maxsulotni ishlab chiqarish uchun sarf qilingan xarajatni (uni y bilan belgilaymiz) maxsulot miqdori x ga bog'liq deb hisoblash mumkin. Ya'ni $y = f(x)$, *bu funksiya ishlab chiqarish funksiyasi deyiladi.*

Ishlab chiqarilayotgan maxsulot miqdori Δx ga o'zgarsin, bu holda ishlab chiqarish xarajati ham o'zgaradi va u $\Delta y = f(x_0 + \Delta x) - f(x_0)$ bo'ladi. Ishlab chiqarish xarajati orttirmasining ishlab chiqaradigan maxsulot orttirmasiga nisbatani olamiz:

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (1)$$

(1) da limitga o'tamiz:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (2)$$

(2) limit yoki $f(x)$ funksiyaning hosilasi, iqtisodiyotda ishlab chiqarishning limitik xarajatlari

deyiladi (*ishlab chikarish xarajatlarining uzgarish tezligi*).

***Izoh*:** Korxona tomonidan biror maxsulot ishlab chikarish xarajatlari, odatda ikki xil bo'lishi ko'zda tutiladi:

a) o'zgaruvchi xarajatlar, bular korxona ishlab chiqaradigan maxsulotlar hajmiga proporsional bo'lib, xomashyo va uni keltirishga stanoklar iste'mol qiladigan elektr energiyaga, ishchilarga beriladigan ish haqiga va boshqalarga ketadigan mablag'lardan qushiladi;

b) doimiy xarajatlar, bular asosan ishlab chiqariladigan maxsulot xajmiga bog'liq bo'lmaydigan xarajatlardir. Bunga imoratlar amartizasiyasi, ba'zi kategoriyadagi yordamchi ishchi va xizmatchilar ish xaqi, imoratlarni yoritish va isitishga hamda boshqalarga sarf qilinadigan xarajatlar kiradi.

5. Funksiyaning differensiallanuvchanligi va uzluksizligi orasidagi bog'lanish

Funksiyaning hosilaga ega bo'lish xususiyati, uning uzluksiz bo'lish xususiyati bilan chambarchas bog'langan.

Teorema: ***Berilgan nuqtada hosilaga ega bo'lgan y = f(x) funksiya shu nuqtada uzluksizdir.***

Isboti: Funksiya x nuqtada hosilaga ega bo'lsa, Ushbu limit

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$$

mavjud buladi.

Bu holda

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)+\alpha(\Delta x) \Rightarrow$$

$$\Delta y = f(x+\Delta x) - f(x) = f'(x)\Delta x + \alpha(\Delta x)\Delta x, \quad (1)$$

bu yerda $\lim\limits_{\Delta x\to 0}\alpha(\Delta x) = 0.$

Agar $\Delta x \to 0$, (1) dan $\Delta y \to 0$ bo'lishini ko'rish oson, bu esa ta'rifga asosan $y = f(x)$ funksiyaning x nuqtada uzluksizligini bildiradi.

Izoh: *Teskari teorema noo'rindir. Alohida olingan nuqtalarda differensiallanuvchi bo'lmagan uzluksiz fuknsiyalar ham mavjud.*

Masalan, $y=\sqrt[3]{x^2}$ funksiya barcha $x\in R$ nuqtalarda uzluksiz, biroq u $x=0$ nuqtada differensiallanuvchi emas. Haqiqatan ham, $x=0$ nuqtada

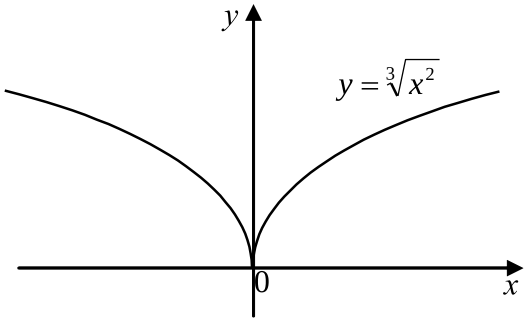
$$\Delta y = \sqrt[3]{(0+\Delta x)^2} - \sqrt[3]{0} = \sqrt[3]{(\Delta x)^2};$$

$$\frac{\Delta y}{\Delta x} = \frac{\sqrt[3]{(\Delta x)^2}}{\Delta x} = \frac{1}{\sqrt[3]{\Delta x}}.$$

limitga o'tib quyidagini hosil qilamiz:

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\sqrt[3]{\Delta x}} = \infty$$

ya'ni hosila mavjud emas.



6. Asosiy elementar funksiyalarni differensiallash

1. Trigonometrik funksiyalarni differensiallash

a) $y=\sin x,\ x\in R$ funksiyani qaraymiz. x ga Δx orttirma bersak y funksiya Δy orttirma oladi, bu holda

$$\Delta y=\sin(x+\Delta x)-\sin x=2\cdot\cos\frac{(x+\Delta x)+x}{2}\cdot\sin\frac{\Delta x}{2}=2\cdot\cos(x+\frac{\Delta x}{2})\cdot\sin\frac{\Delta x}{2}$$

$$\frac{\Delta y}{\Delta x}=\frac{2\cos(x+\frac{\Delta x}{2})\cdot\sin\frac{\Delta x}{2}}{\Delta x}=\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\cos(x+\frac{\Delta x}{2})$$

$$y'=\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}\cdot\lim_{\Delta x\to0}\cos(x+\frac{\Delta x}{2})=1\cdot\cos x=\cos x.$$

Shunday qilib, $(\sin x)'=\cos x.$

b) $y=\cos x,\quad x\in R$ funksiyani qaraymiz. Ma'lumki, keltirish formulasidan: $\cos x=\sin\left(x+\frac{\pi}{2}\right)$, endi $x+\frac{\pi}{2}=u$ deb, murakkab funksiyaning hosilasi qoidasidan foydalanamiz:

$$u'_x=(x+\frac{\pi}{2})'=1;\qquad y'_x=(\sin u)'_u\cdot u'_x=\cos u\cdot u'_x=\cos(x+\frac{\pi}{2})\cdot1=-\sin x.$$

Shunday qilib, $(\cos x)'=-\sin x.$

v) $y=tgx,\quad x\in R$ funksiyani qaraymiz. $tgx=\frac{\sin x}{\cos x}$ bo'lgani uchun kasrning hosilasi qoidasidan foydalansak

$$y'=(tgx)'=\left(\frac{\sin x}{\cos x}\right)'=\frac{(\sin x)'\cdot\cos x-(\cos)'\cdot\sin x}{\cos^2x}=\frac{\cos x\cdot\cos x+\sin x\cdot\sin x}{\cos^2x}=\frac{1}{\cos^2x};$$

demak $(tgx)'=\frac{1}{\cos^2x}$ ni olamiz.

Xuddi shunday, $(ctgx)'=-\frac{1}{\sin^2x}$ formula isbotlanadi.

2. Logarifmik va kursatkichli funksiyani differensiallash

a) $y = \ln x,\ 0 < x < \infty$ funksiyani qaraymiz. x ga Δx orttirma berib, $\frac{\Delta y}{\Delta x}$ nisbatni hisoblaymiz:

$$\frac{\Delta y}{\Delta x} = \frac{\ln(x+\Delta x) - \ln x}{\Delta x} = \frac{\ln\frac{x+\Delta x}{x}}{\Delta x} = \frac{1}{x}\cdot\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}}$$

Δx ni 0 ga intiltirib limitga o'tamiz:

$$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{x}\cdot\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = \frac{1}{x}\cdot\lim_{\Delta x\to 0}\frac{\ln\left(1+\frac{\Delta x}{x}\right)}{\frac{\Delta x}{x}} = \frac{1}{x}\cdot 1.$$

Shunday qilib $(\ln x)' = \frac{1}{x}.$

Endi,

$$y = \log_a x \Rightarrow\quad y = \frac{\ln x}{\ln a} \Rightarrow\quad y' = \left(\frac{\ln x}{\ln a}\right)' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x\ln a}.$$

Shunday qilib, $(\log_a x)' = \frac{1}{x\ln a}.$

b) $y = a^x,\quad 0 < a \neq 1,\quad x \in R.$

Oshkormas funksiya hosilasini topish qoidasidan foydalanamiz:

$y = a^x,\ \Rightarrow\ \ln y = x\ln a.$

Tenglikning har ikkala tomonini argument x bo'yicha differensial-laymiz: $(\ln y)' = \frac{y'}{y};\quad \frac{1}{y}\cdot y' = \ln a \Rightarrow\quad y' = y\ln a \Rightarrow\quad y' = a^x\ln a \Rightarrow\quad (a^x)' = a^x\ln x.$

Xususiy holda, agar $a = e$ bo'lsa $(e^x)' = e^x.$

3. Darajali funksiyani differensiallash

$y = x^\alpha$ darajali funksiya, $\alpha \in R$ bo'lganda $x > 0$ uchun qaraladi. Bu holda $x^\alpha = e^{\alpha\ln x}$. Murakkab funksiya hosilasini topish qoidasini qo'llaymiz:

$$(x^\alpha)' = (e^{\alpha\ln x})' = e^{\alpha\ln x}(\alpha\ln x)' = e^{\alpha\ln x}\cdot\alpha\cdot\frac{1}{x} = x^\alpha\cdot\alpha\cdot\frac{1}{x} = \alpha x^{\alpha-1}.$$

Shunday qilib, $(x^\alpha)' = \alpha x^{\alpha-1}.$

Misollar:

Funksiyalarning hosilalari topilsin:

a) $y = x$; bu xolda $y' = (x)' = 1\cdot x^{1-1} = 1\cdot x^0 = 1;$

b) $y = x^{50}$; bu xolda $y' = (x^{50})' = 50\cdot x^{50-1} = 50\cdot x^{49};$

v) $y = \sqrt[5]{x}$; bu xolda $y' = (\sqrt[5]{x})' = \left(x^{\frac{1}{5}}\right)' = \frac{1}{5}x^{\frac{1}{5}-1} = \frac{1}{5\cdot\sqrt[5]{x^4}};$

g) $y = x^{\sqrt{3}}$; bu xolda $y' = \left(x^{\sqrt{3}}\right)' = \sqrt{3}\cdot x^{\sqrt{3}-1}$;

d) $y = \sqrt[8]{x^7}$; bu xolda $y' = (\sqrt[8]{x^7})' = \left(x^{\frac{7}{8}}\right)' = \frac{7}{8}x^{\frac{7}{8}-1} = \frac{7}{8\sqrt[8]{x}}$;

e) $y = x^8 + 3x^{15} + 20x^{19}$; bundan $y' = 8x^7 + 45x^{14} + 380x^{18}$;

j) $y = 19x^6 + \sqrt[3]{x} - \frac{9}{x^{30}}$; bundan $y' = 114x^5 + \frac{1}{3}x^{-\frac{2}{3}} + 270x^{-31}$.

7. Teskari funksiya hosilasi

y = f (x) , x є D(f) , y є E(f) funksiyani qaraymiz. Ma'lumki agar har bir *y є E(f)* uchun yagona *x є D(f) q*iymat aniqlanganda ham biz funksiyaga ega bo'lamiz. Bu funksiya *f (x)* ga nisbatan teskari funksiya deyiladi va $x = f^{-1}(y)$ deb belgilanadi. O'z navbatida *y = f (x)* funksiya $x = f^{-1}(y)$ uchun teskari funksiya bo'ladi. Shu sababli har ikkala funksiya o'zaro teskari funksiya bo'ladi.

y = f (x) funksiya ixtiyoriy *x* nuqtada $f'(x) = \lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x}$ hosilaga ega va $f'(x) \neq 0$ bo'lsin. $x = f^{-1}(y)$ teskari funksiyaning mos *y = f (x)* nuqtadagi hosilasini topish uchun $\lim\limits_{\Delta y\to 0}\frac{\Delta x}{\Delta y} = (f^{-1}(y))'$ limitni topamiz.

Funksiyaning uzuksizligi natijasida $\Delta x \to 0 \Rightarrow \Delta y \to 0$
Shu sababli,

$$\lim_{\Delta y\to 0}\frac{\Delta x}{\Delta y} = \lim_{\Delta x\to 0}\frac{1}{\frac{\Delta y}{\Delta x}} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)^{-1} = \left[f'(x)\right]^{-1} = \frac{1}{f'(x)},$$

ya'ni $[f'(x)]^{-1} = \frac{1}{f'(x)}$ bunda $f'(x)$ hosila $x = f^{-1}(y)$ nuqtada hisobla-nadi.

Xulosa: O'zaro teskari funksiyalarning hosilalari miqdori bo'yicha teskaridir.

Teskari trigonometrik funksiyalarni differensiallash

a) *Arksinus va arkosinuslarning hosilalari.*

$y = \arcsin x,\quad x\in[-1;1]$ funksiyani qaraymiz.

$y = \arcsin x,\quad x\in[-1;1]$ va $x = \sin x,\quad y\in[-\frac{\pi}{2};\frac{\pi}{2}]$ funksiya o'zaro teskari funksiyalardir. Shu sababli, teskari funksiya hosilasini topish qoidasidan foydalanamiz:

$$y'_x = \frac{1}{x'_y} = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Shuni aytish kerakki, bu yerda $(\cos y) > 0,\quad y\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right)$

Demak, $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$.

Ma’lumki, $\arcsin x+\arccos x=\frac{\pi}{2}$, ya’ni $\arccos x=\frac{\pi}{2}-\arcsin x$.

Bundan: $(\arccos x)'=-(\arcsin x)'=-\frac{1}{\sqrt{1-x^2}}$.

Shunday kilib $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}}, \quad x\in(-1;1)$.

b) *Arktangens va arkkotangensning xosilalari.*

$y=arctgx$ funksiyani kraymiz. U $x=tg\,y$ uchun teskari funksiyadir. Shu sababli

$$x'(y)=(tgy)'=\frac{1}{\cos^2 y}=\sec^2 y, \qquad (arctgx)'=\frac{1}{(tgy)'}=\frac{1}{\sec^2 y}.$$

O’z navbatida $sec^2y = 1 + tg^2y = 1 + x^2 \neq 0$, u holda izlangan hosila x ning barcha qiymatlarida mavjud va shunday qilib,

$$(arctgx)'=\frac{1}{1+x^2}.$$

Agar $y=arctg\,u$, $u=u(x)$ murakkab funksiya berilsa, $u = u(x)$ – differensial-lanuvchi funksiya bo’lganda

$$(arctgu)'=\frac{u'}{1+u^2}$$

formulaga ega bo’lamiz.

Xuddi shunday $y=arcctg\,u$ funksiya uchun $(arcctgx)'=-\frac{u'}{1+x^2}$ ni hosil qilamiz.

$y = arcctg\,u$, $u = u(x)$ murakkab funksiya bo’lgan holda $(arcctgu)'=-\frac{1}{1+u^2}$ ga ega bo’lamiz

Misol

1) $$\left(arctg\frac{2x}{3}\right)'=\frac{\left(\frac{2x}{3}\right)'}{1+\left(\frac{2x}{3}\right)^2}=\frac{2}{3\left(1+\frac{4x^2}{9}\right)}=\frac{6}{9+4x^2};$$

2) $$(x\cdot arctg2x)'=1\cdot arctg2x+x\cdot(arctg2x)'=arctg2x+x\cdot\frac{(2x)'}{1+4x^2}=arctg2x+\frac{2x}{1+4x^2};$$

8. Differensiallashning asosiy qoidalari

Hosilani topish algoritmi va funksiya limitining xossalarini qo’llab, quyidagi teoremani isbotlash mumkin.

<u>Teorema:</u> *$u = u(x)$ va $v = v(x)$ funksiyalar ixtiyoriy x nuqtada diffe-rensiallanuvchi bo’lsa, u holda Cu , $u \pm v$, uv , $v \neq 0$ bulganda $\frac{u}{v}$ funksiyalar xam x nuqtada differensiallanuvchi va*

$(C)' = 0;$ $\qquad (u v)' = u'v + u v';$

$(C u)' = C u';$ $\qquad \left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2};$

$(u + v)' = u' + v';$ $\qquad \left(\frac{C}{v}\right)'=-\frac{Cv'}{v^2};$

bu yerda C – const.

Teoremani isbotlash o'quvchiga havola qilinadi.

9. Murakkab funksiya hosilasi

$y = f(u)$, $u = \varphi(x)$, $x \in X$ murakkab funksiya berilgan, x nuqtada $u' = \varphi'(x)$ hosila, $u = \varphi(x)$ nuqtada esa $y'_u = f'(x)$ hosila mavjud bo'lsin, ya'ni:

$$u'_x = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \varphi'(x); \quad y'_u = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} = f'(u) \quad \text{bo'lsin.}$$

Ushbu $\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x}$ ayniyatni tuzamiz.

Bundan:

$$y'_x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left[\frac{\Delta y}{\Delta u}\cdot\frac{\Delta u}{\Delta x}\right] = \lim_{\substack{\Delta x \to 0 \\ (\Delta u \to 0)}} \frac{\Delta y}{\Delta u}\cdot\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} = f'(u)\cdot u'_x;$$

$$y'_x = y'_u \cdot u'_x$$

yoki $$[f(\varphi(x))]' = f'(\varphi(x)) \cdot \varphi'(x)$$

Shunday qilib, quyidagi isbotlandi:

Teorema: *Agar $u = \varphi(x)$ funksiya x nuqtada u'_x hosilaga ega, $y = f(u)$ funksiya esa $u = \varphi(x)$ nuqtada $y'_u = f'(u)$ hosilaga ega bo'lsa, u holda $y = f(\varphi(x))$ murakkab funksiya x nuqtada y'_x hosilaga ega va* $y'_x = y'_u \cdot u'_x$ *formula o'rinlidir.*

Boshqacha qilib aytganda, murakkab funksiyaning hosilasi berilgan funksiyaning oraliq argumenti bo'yicha hosilasini oraliq argumentning asosiy argument buyicha hosilasiga ko'paytirilganiga teng.

Izoh: Oraliq argumentlar soni bir nechta bo'lgan murakkab funksiya hosilasi ham shu yo'sinda topiladi. Masalan, $y = f(x)$, $u = \varphi(v)$, $v = \psi(x)$ bo'lsin. Bu holda $y'_x = y'_u \cdot u'_v \cdot v'_x$.

9. Oshkormas funksiyalarni differerensiallash

Agar $y = f(x)$ funksiya y ga nisbatan yechilmagan $F(x,y) = 0$ (1) tenglama bilan berilgan bo'lsa, ma'lum shartlar bajarilganda, bunday bog'lanish y ni x ning oshkormas funksiyasi ko'rinishida aniqlaydi deb ataladi. Bu holda $y = f(x)$ qiymatni tenglamaga qo'yganda, tenglama ayniyatga aylanishi kerak: $F(x, f(x)) \equiv 0$.

Oshkormas funksiyani differensiallash shunga asoclanganki, x ni o'z ichiga olgan har qanday ayniyatning hosilasi, argumentga nisbatan ayniyat bo'ladi. Agar x argument va uning y funksiyasi orasidagi munosabat y ga nisbatan yechilmagan tenglama bilan berilgan bo'lsa, bu holda y ning x ga nisbatan hosilasini topish uchun *y ni x ning funksiyasi deb qarab,* tenglamani x ga nisbatan differensiallash kerak. Shu yo'l bilan olingan va x, y, y' ni o'z ichiga olgan tenglamani y'ga isbatan yechib, y'ning qiymati x va u orqali topiladi.

1- misol

Funksiya $x^2 + y^2 = 3$ tenglama bilan berilgan. $\left(-\sqrt{2};\sqrt{2}\right)$ nuqtada y' topilsin.

Yechilishi: Tenglamani x buyicha differensiallaymiz:

$$2x + 2y \cdot y' = 0, \qquad y' = -\frac{x}{y}.$$

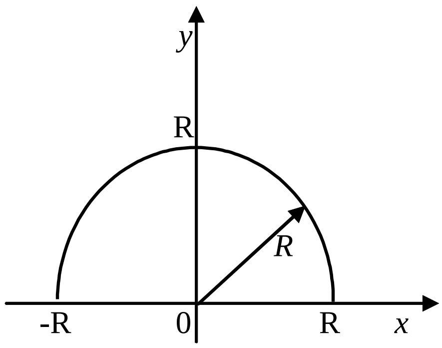
Endi, bunga $x=-\sqrt{2}$, $y=\sqrt{2}$ ni qo'yib, $y'(-\sqrt{2})=-\dfrac{-\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}}{\sqrt{2}}=1$ ni olamiz.

Demak, berilgan aylanaga $\left(-\sqrt{2};\sqrt{2}\right)$ nuqtada o'tkazilgan urinma *Ox* o'qi bilan $\alpha=\dfrac{\pi}{4}$ $(tg\alpha = 1)$ burchak hosil qiladi.

10. Parametrik ko'rinishda berilgan funksiya hosilasi

$x = \varphi(t)$ va $y = \psi(t)$ funksiyalar t_0 nuqtaning biror atrofida aniqlangan va ulardan biri, masalan, $x = \varphi(t)$ qayd qilingan atrofda $t = \varphi^{-1}(x)$ teskari funksiyaga ega bo'lsin. Bu holda $y(\varphi^{-1}(x))$ murakkab funksiya $x = \varphi(t)$, $y = \psi(t)$ formulalar yordamida parametrik berilgan deyiladi (t - parametr). Masalan, $x = R\cos t$ va $y = R\sin t$ $0 \le t \le \pi$, $R>0$ funksiyalar $y=\sqrt{R^2-x^2}$ funksiyaning parametrik ko'rinishi bo'ladi. Bu funksiya grafigi aylananing yukori yarim tekislikda yotgan qismini ifodalaydi *(2-shakl)*.

Endi, $x = \varphi(t)$ va $y = \psi(t)$ funksiya-lar differensiallanuvchi va $x'(t)\neq 0$ bo'lsin. $\dfrac{dy}{dx}$ hosilani topamiz. Murakkab funksiyani differensiallash qoidasiga asosan $\dfrac{dy}{dx}=\dfrac{dy}{dt}\dfrac{dt}{dx}$ bo'ladi.



2-shakl

$\dfrac{dt}{dx}=\dfrac{1}{\dfrac{dx}{dt}}$ bulgani uchun, bu holda $\dfrac{dy}{dx}=\dfrac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$ (1) bo'ladi. (1) formulani

$y'_x=\dfrac{y'_t}{x'_t}$ (1') ko'rinishda yozish mumkin.

Misol

Parametrik ko'rinishda berilgan $x=\sqrt{t}$; $y=t^2$ funksiyaning $\dfrac{dy}{dx}$ hosilasi topilsin.

Yechilishi: $y'_t=(t^2)'=2t$; $x'_t=(\sqrt{t})'=\dfrac{1}{2\sqrt{t}}$, (1) formulaga asosan

$$y'_x=\frac{y'_t}{x'_t}=\frac{2t}{\dfrac{1}{2\sqrt{t}}}=4t\sqrt{t}.$$

3-ilova

I.2. Berilgan mazmunga qarab “Funksiya hosilasi ” mavzusi bo‘yicha talabalar bilishi lozim bo‘lgan savollar

Savollar	Javoblar
1.Hosilaga tushunchasiga keltirila-digan masalalardan, **oniy tezlik** haqidagi masala qanday bo’ladi?	1. Fizikadan ma’lumki, bo’shliqda moddiy nuqtaning erkin tushishi qonuni $$S = \frac{g}{2}t^2 \qquad (1)$$ munosabat bilan ifodalanib, bu yerda t erkin tushish boshlanishidan hisoblangan vaqt, S t vaqtda o’tgan yo’l, g erkin tushish tezlanishi, $g \approx 9{,}81ì\ /ñåê^2$. Bu harakat notekis bo’lib, uning tezligini topish masalasini qaraymiz. Vaqtning biror aniq t momenti (oni)ni qaraylik. Bu momentda moddiy nuqta A holatda bo’lsin. OA yo’lning miqdori (1) formula bilan topiladi. Vaqt Δt miqdorga ortsin, ya’ni t, Δt orttirma qabul qiladi. $t+\Delta t$ momentda nuqta B holatda bo’ladi. AB, vaqt Δt orttirma olgandagi yo’l orttirmasi, uni $AB = \Delta S$ bilan belgilaymiz. (1) formulaga $t+\Delta t$ qo’yib, $S + \Delta S = \frac{g}{2}(t+\Delta t)^2$, bundan $\Delta S = \frac{g}{2}(t+\Delta t)^2 - \frac{gt^2}{2}$ yoki $\Delta S = \frac{g}{2}(2t\Delta t + \Delta t^2)$. Oxirgi tenglikni Δt ga bo’lib, $$\frac{\Delta S}{\Delta t} = \frac{g}{2}(2t + \Delta t) \qquad (2)$$ natijani olamiz. Oxirgi tenglikdan ma’lumki, $\Delta S / \Delta t$ nisbat t va Δt ga bog’liq. Shuning uchun, notekis harakatning tezligi faqat vaqtning aniq momentiga tegishli bo’ladi. Shunday qilib, vaqtning har bir momentidagi ***<u>oniy tezlik</u>*** <u>topish</u> masalasi kelib chiqadi. (2) tenglikdan ma’lumki, t o’zgarmas bo’lganda, $\Delta S / \Delta t$ A dan B holatgacha oraliqdagi o’rtacha tezlik bo’lib, uni v_{yp} bilan belgilaymiz. Ma’lumki, (2) da Δt qancha kichik bo’lsa, t momentdagi tezlikni shuncha yaxshiroq ifodalaydi. Bundan shunday xulosaga kelamizki, erkin tushayotgan nuqtaning t momentidagi oniy tezligi v ni $v_{óð}$ o’rtacha tezlikning $\Delta t \to 0$ dagi limiti kabi aniqlaymiz, ya’ni $v = \lim_{\Delta t \to 0} v_{óð}$ Shunday qilib, oniy tezlikni hisoblash uchun qo’yidagi ko’rinishdagi limitni hisoblash kerak bo’ladi. $$v = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} \qquad (3)$$

	(3) ko'rinishdagi limitni hisoblashga ko'p sondagi amaliy masalalarni yechishda to'g'ri keladi.
2. Funksiya hosilasi ta'rifi nimadan iborat?	2. ta'rif. $y = f(x)$ funksiya (a, b) intervalda aniqlangan bo'lib, x_0 nuqtadagi funksiya Δy orttirmasining Δx argument orttirmasiga nisbatining, argument orttirmasi nolga intilgandagi limitiga, $y = f(x)$ funksiyaning x_0 nuqtadagi <u>hosilasi</u> deyiladi. Bu limit $$y', f'(x_0), \ \frac{dy}{dx}, \ \frac{df}{dx}$$ simvollardan biri bilan belgilanadi. Ta'rifga asosan $$f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta ó}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$ bo'ladi, bu limit mavjud bo'lsa, hosila x_0 nuqtada mavjud deyiladi.
3. $y = x^3$ funksiyaning hosilasini hosila ta'rifiga asosan toping.	3. $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ limitni hisjoblaymiz: $$\Delta y = (x + \Delta x)^3 - x^3 = x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3 =$$ $$= 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3$$ bo'lib, $$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x} =$$ $$= \lim_{\Delta x \to 0} \frac{\Delta x(3x^2) + 3x\Delta x + \Delta x^2}{\Delta x} =$$ $$= \lim_{\Delta x \to 0}(3x^2 + 3x\Delta x + \Delta x^2) = 3x^2.$$ bo'ladi. Shunday qilib, $y' = 3x^2$.
4.Hosilaning geometrik ma'nosi nimadan iborat?	4. Hosila muhim geometrik ma'noga ega. Bu funksiyaning x_0 nuqtadagi hosilasi uning grafigiga $M(x_0, f(x_0))$ ***<u>nuqtada o'tkazilgan urinmaning</u>*** OX o'qining musbat yo'nalishi bilan hosil qilgan burchagining tangensiga teng. $y = f(x)$ egri chiziqqa $M_0(x_0, y_0)$ nuqtadan ***<u>o'tkazilgan urinma tenglamasi</u>*** $$y - y_0 = f'(x_0)(x - x_0)$$ bo'ladi, bunda $y_0 = f(x_0)$.
5.Murakkab funksiya hosilasi qanday aniqlanadi?	5. $y = f(u)$, $u = \varphi(x)$, ya'ni $y = f[\varphi(x)]$ **<u>murakkab funksiya</u>** bo'lsa, $y = f(u)$ funksiyaning x o'zgaruvchi bo'yicha hosilasi $y' = f'(u) \cdot u'$ bo'ladi.
6.Differensiallash qoidalari qanday edi?	6. x erkli o'zgaruvchi, $u = u(x)$ va $v = v(x)$ uning differensiallanuvchi funksiyalari bo'lsin. 1.,$\tilde{N}' = 0$ $C-$o'zgarmas miqdor. 2. $x' = 1$.

	3. $(u \pm v)' = u' \pm v'$. 4. $(u \cdot v)' = u'v + uv'$. 5. $(cu)' = c \cdot u'$. 6. $\left(\frac{u}{v}\right)' = \frac{u'v - u \cdot v'}{v^2}$.
7. Murakkab funksiya uchun hosilalar jadvali qanday bo'ladi?	7. **Murakkab funksiya uchun hosilalar jadvali** quyidagicha bo'ladi: 1) $(u^n)' = nu^{n-1} \cdot u' \quad n \in R, \quad u > 0$; 2) $(a^u)' = a^u \cdot 1na \cdot u'$; 3) $(e^u)' = e^u \cdot u'$; 4) $(\log_a u)' = \frac{1}{u \cdot 1na} \cdot u'$; 5) $(\ln u)' = \frac{1}{u} \cdot u'$; 6) $(\sin u)' = \cos u \cdot u'$; 7) $(\cos u)' = -\sin u \cdot u'$; 8) $(tg\, u)' = \frac{1}{\cos^2 u} \cdot u'$; 9) $(ctg u)' = -\frac{1}{\sin^2 u} \cdot u'$; 10) $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$; 11) $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$; 12) $(arctg\, u)' = \frac{1}{1+u^2} \cdot u'$; 13) $(arcctg\, u)' = -\frac{1}{1+u^2} \cdot u'$; 14) $(u^v)' = vu^{\grave{i}-1} \cdot u' + u^v \cdot 1nu \cdot v'$.
<u>8. Oshkormas ko'rinishda berilgan funksiyalarning hosilasiqanday bo'ladi?</u>	8. x o'zgaruvchining y funksiyasi <u>***oshkormas ko'rinishda***</u> $F(x, y) = 0$ berilgan bo'lsa, y' hosilani topish uchun $F(x, y) = 0$ tenglikni x bo'yicha differensiallab, so'ngra hosil bo'lgan tenglamadan y' ni topamiz. Ikkinchi va undan yuqori tartibli hosilalar ham shu kabi topiladi.
<u>10. Parametrik ko'rinishda berilgan funksiyaning hosilasi qnday topiladi?</u>	10. Funksional bog'lanish <u>parametrik</u> $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ ko'rinishda berilgan bo'lsa, $dy/\ dx,\ d^2y/\ dx^2$ hosilalar
11. $y = \frac{x^3}{3} + 4$ egri chiziqqa abssissasi	

$x_0 = 2$ nuqtada o'tkazilgan urinma va normalning tenglamasini yozing. 14. $x^2 + y^2 = 100$ oshkormas ko'rinishda berilgan, *ó* funksiyaning hosilani toping. 15. $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ parametrik ko'rinishda berilgan, y funksiyaning ikkinchi tartibli hosilasini toping.	$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}\cdot\frac{dx}{dt} - \frac{d^2x}{dt^2}\cdot\frac{dy}{dt}}{\left(\frac{dx}{dt}\right)^3}$ formula bilan topiladi. 11. $y_0 = \frac{20}{3}$, $y'(2) = 2^2 = 4$, $y - \frac{20}{3} = 4(x-2)$ yoki $3y - 20 = 12(x-2)$, $12x - 3y - 4 = 0$, bu $M_0(2{,}20/3)$ nuqtadan o'tkazilgan urinmaning tenglamasi. Normalning burchak koeffisiyenti $-\frac{1}{f'(x_0)} = -\frac{1}{4}$, demak, $y - \frac{20}{3} = -\frac{1}{4}(x-2)$ yoki $12y - 80 = -3(x-2)$, $3x + 12y - 86 = 0$ bo'lib, bu M_0 nuqtadan o'tkazilgan normalning tenglamasi bo'ladi. 14. $2x + 2y\cdot y' = 0$; $2yy' = -2x$, $y' = -x/y$; 15. $\frac{dx}{dt} = -a\sin t$, $\frac{d^2x}{dt^2} = -a\cos t$, $\frac{dy}{dt} = a\cos t$, $\frac{d^2y}{dt^2} = -a\sin t$ (1) formulaga asosan, $\frac{d^2y}{dx^2} = \frac{-a\sin t\cdot(-a\sin t) - (-a\cos t)\cdot a\cos t}{(-a\sin t)^3} =$ $= \frac{a^2\sin^2 t + a^2\cos^2 t}{-a^3\sin^3 t} =$ $= -\frac{a^2(\sin^2 t + \cos^2 t)}{a^3\sin^3 t} = -\frac{1}{a\sin^3 t}$. Demak, $\frac{d^2y}{dx^2} = -\frac{1}{a\sin^3 t}$ bo'ladi.